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THE FIRM'S INVESTMENT POLICY UNDER A CONCAVE ADJUSTMENT COST FUNCTION*

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ABSTRACT

The purpose of this article is to examine the effects of a concave adjustment cost function on the optimal dynamic investment policy of a firm. Such an assumption facilitates the explanation of stepwise investment expenditures instead of continuous investments. Therefore, an optimal control model is formulated which allows discontinuities in the level of capital good stock. Using the conditions of the optimal solution we will design a search procedure which enables us to develop the optimal investment pattern.

1. INTRODUCTION

In the literature, adjustment costs within dynamic investment models nearly always are convex functions of investments. This implies rising marginal costs compared to the rate of investment. In that case adjustment costs are minimized through spreading out investment expenditures as much as possible over time. Investments are a smoothing problem.

In this contribution we will introduce a concave adjustment cost function. Such costs imply decreasing marginal costs of investments and therefore it is optimal for the firm to invest either very much or nothing at all. Investments now become a scaling problem.

We will formulate an optimal control model that allows discontinuities in the development of capital good stock at those moments when the large investment expenditures take place. To solve this model, we combine the necessary conditions based on Pontryagin's maximum-principle (see e.g. Kamien and Schwartz [1]) with some additional "jump" conditions, which have been designed by Seierstad and Sydsaeter [2].

From the optimal solution we infer a search procedure that helps us to fix the optimal points of time to invest, as well as the optimal scales of the investment expenditures at the different points of time. The same kind of search procedure was applied by Luhmer [3] in order to solve an inventory problem.

Section 2 contains a short description of the theory of adjustment costs with emphasis on the concave form and its implications. In Section 3 our dynamic model with concave adjustment costs is presented, whereas Section 4 contains the derivation and a further description of the search procedure. Finally, in Section 5 we apply this search procedure to a numerical experiment.

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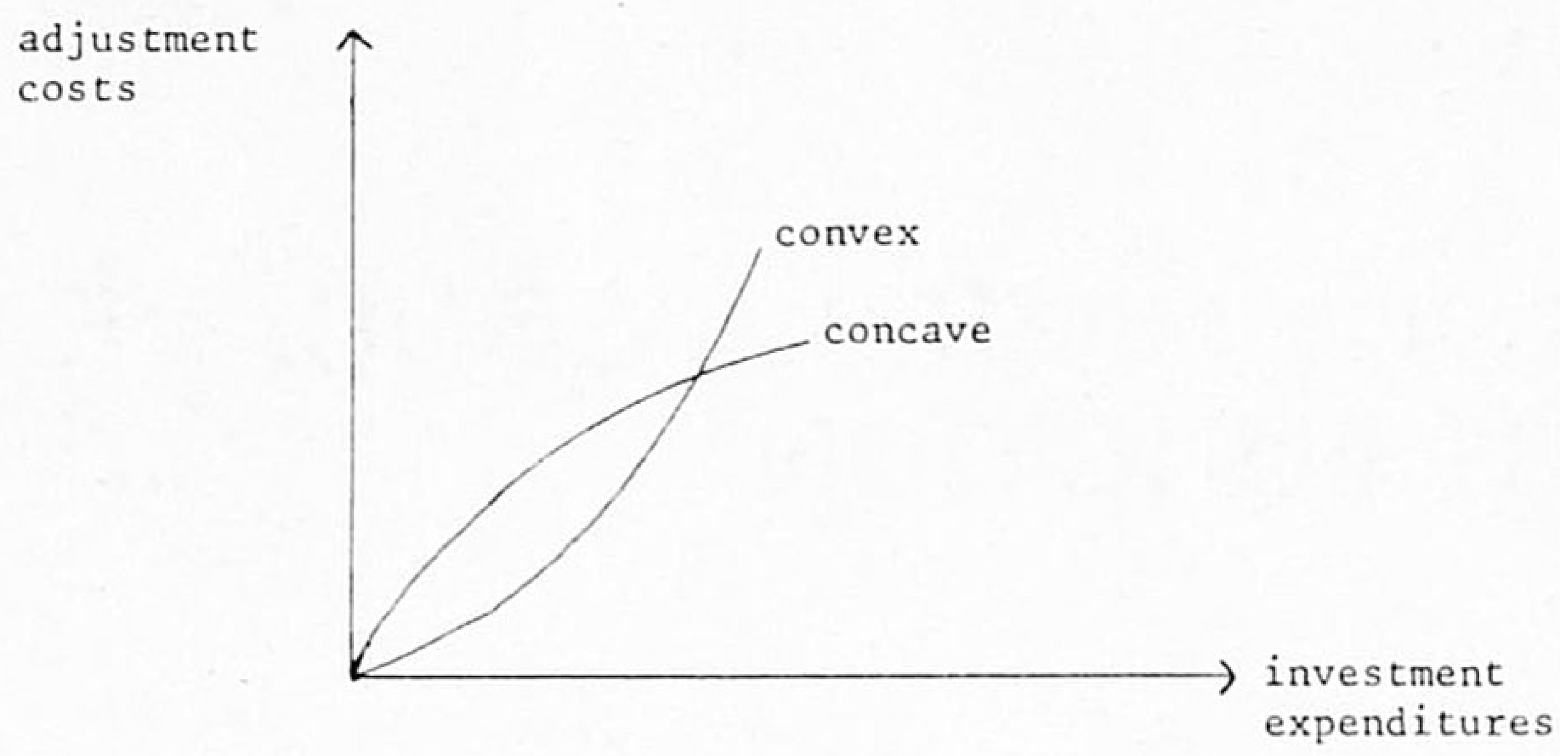

Fig. 1. A convex and a concave adjustment cost function.

2. THE THEORY OF ADJUSTMENT COSTS

In the literature a distinction is made between convex and concave adjustment cost functions (see Fig. 1). Convex adjustment costs apply to a monopsonistic market of capital goods: if the firm wants to increase its rate of growth it will be confronted with increasing prices on the market because of its increased demand of capital goods. Because convex adjustment costs imply rising marginal costs, large investment expenditures are very expensive. Therefore, the firm will tend to adjust its capital good stock slowly instead of instantaneously; investments are a smoothing problem.

In the literature most models have incorporated such a convex adjustment cost function. Some authors, however, like Nickell [4] and Rothschild [5], have argued that there are important economic reasons which plead for a concavely shaped adjustment cost function, such as indivisibilities, use of information, fixed costs of ordering and quantity discounts. In order to illustrate the first two arguments we give two quotations of Rothschild [5]:

"Training involves the use of information (once one has decided how to train one worker, one has in effect decided how to train any number of them), which is a classic cause of decreasing costs. Furthermore, the process is subject to some indivisibilities. It requires at least one teacher to train one worker. Presumably no more teachers are required to train two or three workers."

"Similarly, reorganizing production lines involves both the use of information as a factor of production — (once one has decided how to reorganize one production line, one has figured out how to reorganize two, three or n), and indivisibilities — (one may not be able to reorganize only half or a tenth of a production line)."

If the adjustment cost function is concave, marginal costs are decreasing with increasing investment expenditures. Therefore, the firm minimizes its adjustment costs if its investment policy consists of an alternation of very large investment expenditures and zero investment expenditures. In this way an impulse pattern arises which causes discontinuities in the development of capital good stock. Accordingly, we incorporate concave adjustment costs in an optimal control model that allows discontinuities in the state variable.

3. THE MODEL

We first assume that the firm behaves as if it maximizes its value for the shareholders. This value is expressed as the value of the profits over the planning period plus the value of the firm at the planning horizon. The profits in this model are the difference between the present value of the earnings stream and the sum of the present value of investment expenditures and adjustment costs. The final value of the firm equals the present value of the final capital good stock at the end of the planning period. Further, we assume that the firm operates under decreasing returns to scale and that the adjustment costs are a concave function of investments.

The above results in the next goal function:

$$\max_{I_j, j=1,2,\ldots,n} \int_{T=0}^{z} S(K) \exp(-iT)\, dT - \sum_j (I_j + U(I_j)) \exp(-iT_j) + K(z) \exp(-iz) \qquad (1)$$

in which

I_j $=j$'th investment expenditure
T $=$ time
z $=$ planning horizon
K $=$ total amount of capital goods

$S(K)$ = earnings, $S(K)>0, \frac{dS}{dK}>0, \frac{d^2S}{dK^2}<0$

$U(I_j)$ = adjustment costs of j'th investment, $U(I_j)\geqslant 0, \frac{dU}{dI_j}>0, \frac{d^2U}{dI_j^2}<0, U(0)=0$

T_j = point of time of j'th investment

i = discount rate

We also assume that the amount of capital goods will increase by investments and decrease through depreciations, which are proportional to the value of the capital goods. So, we get the next state equation of capital good stock:

$$\dot{K}=\frac{dK}{dT}=-aK \qquad \text{if } T\neq T_j=1,2,\ldots,n. \tag{2}$$

in which

a = depreciation rate

$$K^+(T)-K^-(T)=I_j \qquad \text{if } T=T_j, j=1,2,\ldots,n. \tag{3}$$

in which

$K^+(T)$ = amount of capital goods just after the investment impulse

$K^-(T)$ = amount of capital goods just before the investment impulse.

Investments are irreversible, so:

$$I_j>0 \text{ for } j=1,\ldots,n \tag{4}$$

Finally, we assume a positive value of the capital good stock at $T=0$:

$$K(0)=K_0>0 \tag{5}$$

Now eqns. (1) through (5) form our dynamic investment model with concave adjustment costs. As discontinuity of the state variable K is allowed, it is a non-standard optimal control model. So, besides Pontryagin's maximum-principle we have to apply additional optimality conditions which have to be fulfilled at jump locations. These kind of necessary optimality conditions are described by Seierstad and Sydsaeter [2]. The application to our problem can be found in the next section.

4. OPTIMAL SOLUTION

At the location of an investment impulse, the following equation must hold:

$$I_j=K^+-K^- \tag{6}$$

definitions:

ψ = co-state variable

H = hamiltonian

$g(K^-,I_j,T_j)$ = K^+-K^-

$-h(K^-,I_j,T_j)$ = total cost of the investment expenditure

The additional necessary conditions, developed by Seierstad and Sydsaeter [2] are the following:

at the jump points, it must hold that:

$$\psi^+-\psi^-=-\frac{\partial h}{\partial K}-\psi^+\frac{\partial g}{\partial K} \tag{7}$$

$$\frac{\partial h}{\partial I_j}+\psi^+\frac{\partial g}{\partial I_j}=0 \tag{8}$$

$$H^+-H^--\frac{\partial h}{\partial T}-\psi^+\frac{\partial g}{\partial T}\ \begin{cases}\geqslant 0 & \text{for } T=0\\ =0 & \text{for } T\in(0,z)\\ \leqslant 0 & \text{for } T=z\end{cases} \tag{9}$$

for all T at which there is no jump, it must hold that:

$$\frac{\partial h(K^-,0,T)}{\partial I}+\psi\frac{\partial g(K^-,0,T)}{\partial I}\leqslant 0 \tag{10}$$

From the model of Section 3, we get that the following must hold:

$$h(K^-,I_j,T_j)=-(I_j+U(I_j))\exp(-iT_j) \tag{11}$$

$$g(K^-,I_j,T_j)=I_j \tag{12}$$

Applying the maximum-principle of Pontryagin to the model of Section 3, we obtain the following necessary conditions:

$$H=S(K)\exp(-iT)-\psi aK \tag{13}$$

$$-\dot{\psi}=\frac{dS}{dK}\exp(-iT)-a\psi \tag{14}$$

$$\psi(z)=\exp(-iz) \quad \text{(transversality condition)} \tag{15}$$

After substituting eqns. (11) and (12) in eqns. (7) through (10) we get:
at the jump points, it must hold that:

$$\psi^{+}-\psi^{-}=0 \tag{16}$$

$$-\left(1+\frac{dU}{dI_j}\right)\exp(-iT)+\psi^{+}=0 \tag{17}$$

$$H^{+}-H^{-}-i(U(I_j)+I_j)\exp(-iT) \begin{cases}\geqslant 0 \text{ for } T=0\\ =0 \text{ for } T\in(0,z)\\ \leqslant 0 \text{ for } T=z\end{cases} \tag{18}$$

for all T at which there is no jump, it must hold that:

$$-\left(1+\frac{dU}{dI}(I=0)\right)\exp(-iT)+\psi\leqslant o \tag{19}$$

From eqn. (16) we can conclude that ψ is continuous at every jump point. Due to the insertion of eqn. (13) in eqn. (18) we obtain that at a jump point it must hold that:

$$(S(K^{+})-S(K^{-}))\exp(-iT)-a\psi(K^{+}-K^{-})$$

$$-i(U(I_j)+I_j)\exp(-iT) \begin{cases}\geqslant 0 \text{ for } T=0\\ =0 \text{ for } T\in(0,z)\\ \leqslant 0 \text{ for } T=z\end{cases} \tag{20}$$

After substituting eqns. (6) and (17) in eqn. (20) and dividing this equation by $\exp(-iT)$ we get:

$$S(K^{+})-S(K^{-})-a\left(1+\frac{dU}{dI_j}\right)I_j- \begin{cases}\geqslant 0 \text{ for } T=0\\ =0 \text{ for } T\in(0,z)\\ \leqslant 0 \text{ for } T=z\end{cases} \tag{21}$$

$$i(U(I_j)+I_j$$

If we substitute in the solution of the differential equation, eqn. (14), the transversality condition eqn. (15), we get:

$$\psi(T)=\exp(aT)\int_{t=T}^{z}\frac{dS}{dK}\exp(-(i+a)t)\,dt + \exp(aT)\exp(-(i+a)z) \tag{22}$$

From eqns. (17) and (22) we finally derive that at a jump point it must hold that:

$$1+\frac{dU}{dI_j}=\exp(-(i+a)(z-T))+\int_{t=T}^{z}\frac{dS}{dK}\exp((-i+a)(t-T))\,dt \tag{23}$$

The left-hand side of expression (23) represents the costs of increasing the investment expenditure by one unit; at the right-hand side we find the marginal earnings of investments consisting of the present value of the remaining new equipment at the end of the planning period (the value of the new equipment decreases with depreciation rate "a" during the rest of the planning period) plus the present value of additional sales over the whole period due to this new equipment (the production capacity of this equipment decreases with a rate "a" during the remainder of the planning period). Expression (23) thus means that at all locations of investment impulses, marginal costs of investments must equal marginal earnings. This is easy to understand, because on the optimal production plan the cost of adjustment involved in installing one additional unit of capital good stock must always balance the net gain of the adjustment. For if it does not balance, then either one unit increase or one unit reduction of the investment at that moment will lead to an increase in the present value of the firm.

Equations (6), (21) and (23) together may be exploited for a search procedure in order to obtain the optimal investment pattern. This can be done in a similar way Luhmer [3] established the optimal ordering plan of the inventory problem under consideration. Contrary to Luhmer, we substitute the co-state variable out of the search procedure by solving its differential equation. However, a consequence is that an integral equation arises and therefore, our search procedure starts at z and goes backwards in time, instead of starting at the initial time point and continuing in course of time until the planning horizon is reached.

The search procedure, that is represented by Fig. 2, starts by choosing an arbitrary value of

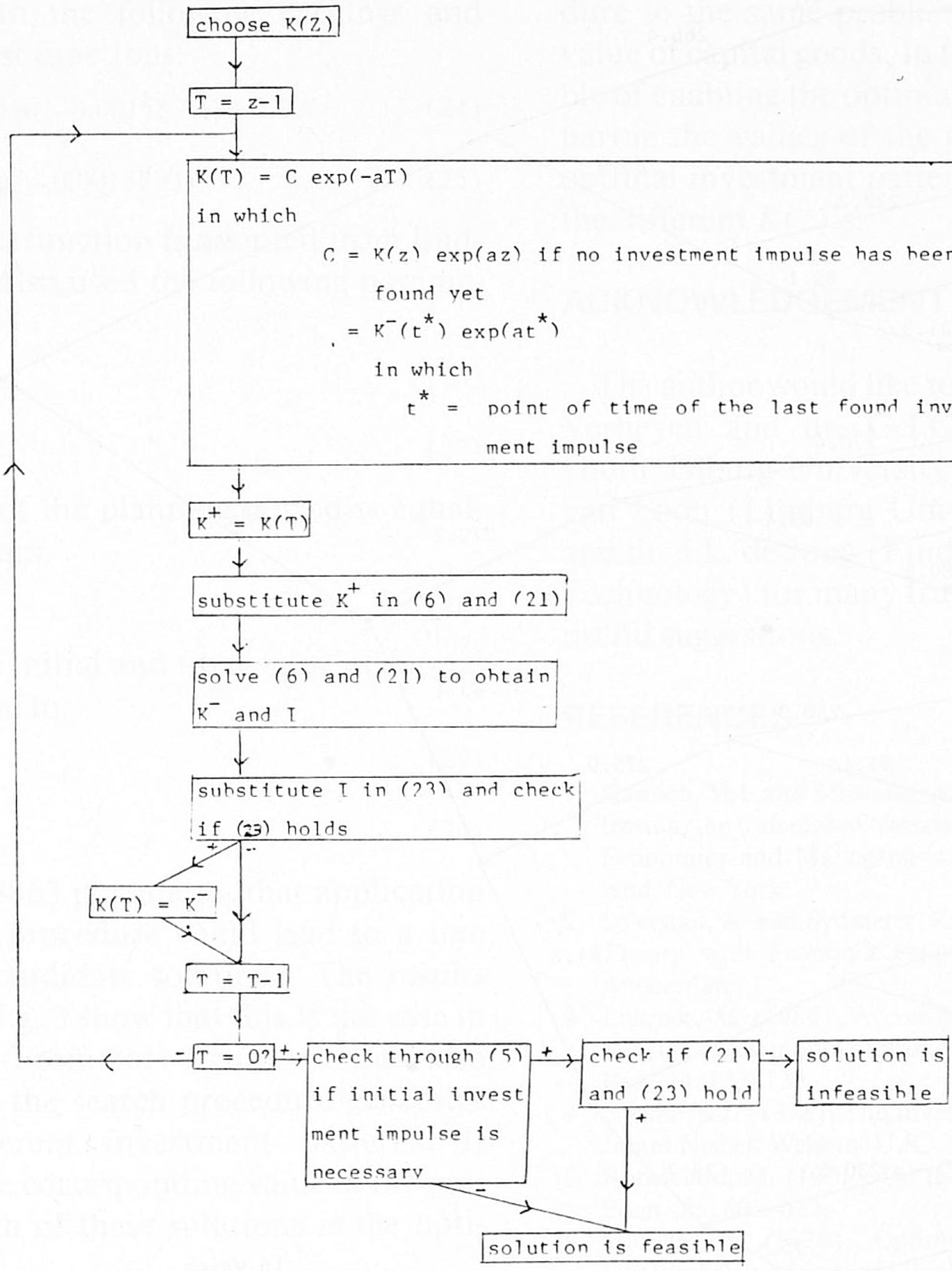

Fig. 2. The search procedure which enables us to develop the optimal investment pattern.

$K(z)$. Obviously, due to eqn. (23) no investment impulse can occur at the planning horizon itself, so we can go immediately to period $T=z-1$. We obtain the magnitude of $K(z-1)$ by substituting $z-1$ for T into the differential equation according to which K behaves during time intervals at which there is no investment impulse. Then we equalize $K(z-1)$ and K^+ and insert this value in eqns. (6) and (21) in order to get the corresponding values of I and K^-. Next, we check whether the obtained value of I fulfills the equality sign of expression (23). In case of an inequality no investment impulse takes place at this point of time; we now go to the previous period and continue the algorithm. If eqn. (23) holds, however, the investment impulse is optimal, $K(z-1)$ becomes equal to K^- and we continue in the same way as before. The algorithm stops as soon as the start of the planning period is reached. From the initial state constraint (5) we can check if an investment impulse is necessary at time point zero. If it is not, the obtained solution is feasible; and if it is, the solution is only feasible

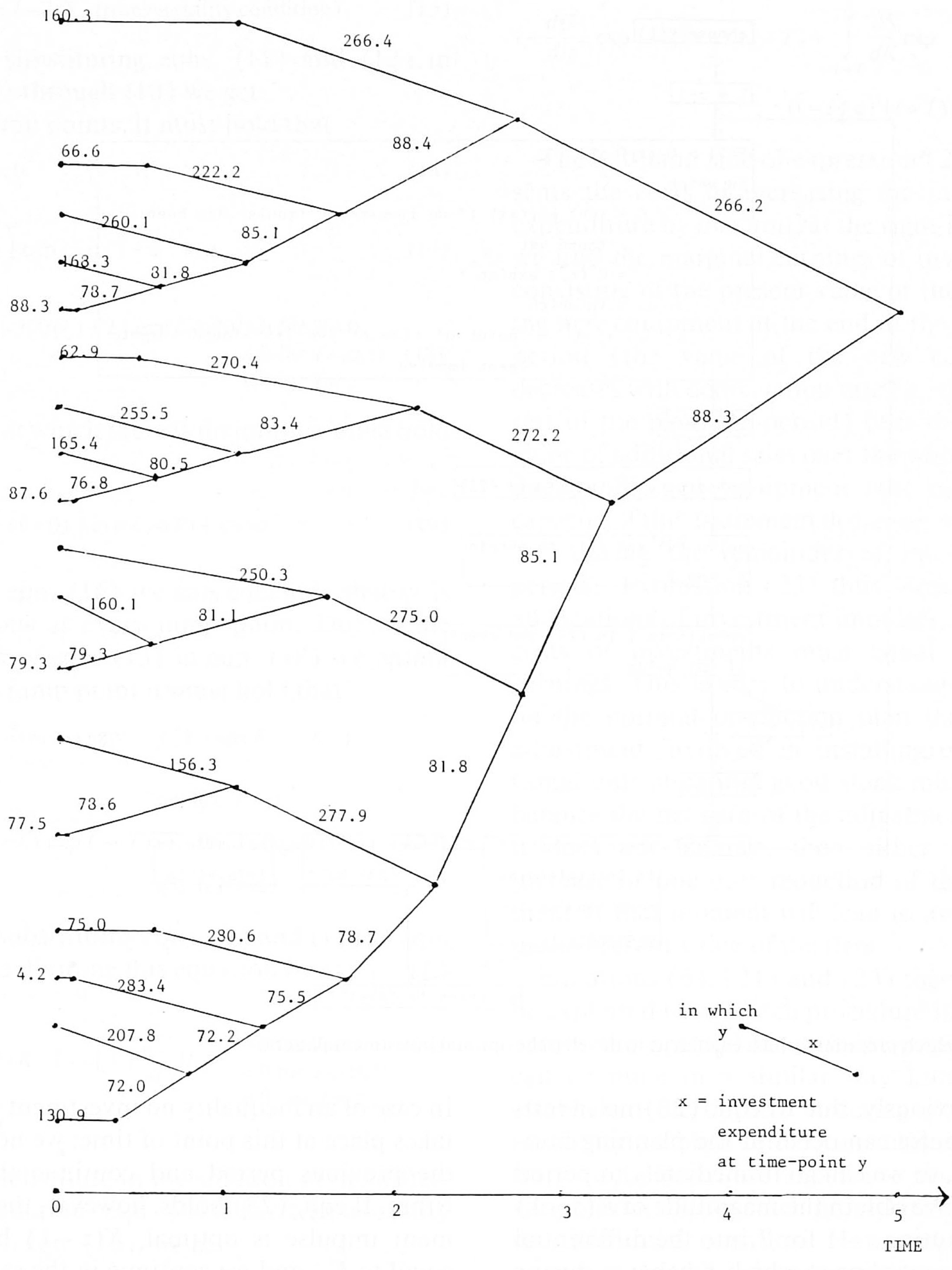

Fig. 3. Candidate solutions of the optimal investment pattern.

when the magnitude of the investment impulse satisfies the inequality sign (!) of eqns. (21) and (23).

By applying this search procedure we can develop investment patterns for every $K(z)$. It depends on the corresponding value of the goal function which of these patterns is the optimal one.

5. NUMERICAL EXPERIMENT

In this paragraph we use the search procedure to derive the optimal investment pattern of a firm with the following earnings and adjustment cost functions:

$$S(K) = 300\ (1 - \exp(-0.00158\ K)) + 0.2\ K \quad (24)$$

$$U(I) = 30\ (1 - \exp(-0.00158\ I)) \quad (25)$$

The earnings function is adopted from Ludwig [6], who also used the following parameter values:

$$i = 0.12 \quad (26)$$

$$a = 0.2 \quad (27)$$

The length of the planning period is equalized to five years:

$$z = 5 \quad (28)$$

Further, the initial and final value of capital goods are equal to:

$$K_0 = 686 \quad (29)$$

$$K(z) = 600 \quad (30)$$

Luhmer (1986) postulates, that application of the search procedure could lead to a tree structure of candidate solutions. The results presented in Fig. 3 show that this is the case in our problem. From this figure, we can also conclude that the search procedure generates eighteen different investment patterns. It depends on the corresponding value of the goal function which of these solutions is the optimal one.

We can, of course, apply the search procedure to the same problem with another final value of capital goods. In this way we are capable of enabling the optimal final value by comparing the values of the goal function of the optimal investment patterns corresponding to the different $K(z)$'s.

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REFERENCES

1 Kamien, M.I. and Schwartz, N. (1983). Dynamic Optimization, the Calculus of Variation and Optimal Control in Economics and Management, 2nd reprint. North Holland, New York.

2 Seierstad, A. and Sydsaeter, K. (1986). Optimal Control Theory with Economic Applications. North Holland, Amsterdam.

3 Luhmer, A. (1986). A continuous time, deterministic, nonstationary model of economc ordering. Eur. J. Oper. Res., 24: 123–135.

4 Nickell, S.J. (1978). The Investment Decisions of Firms. James Nisbet, Welwyn, U.K.

5 Rothschild, M. (1971). On the cost of adjustment. Q. J. Econ., 85: 605–622.

6 Ludwig, T. (1978). Optimale Expansionspfade der Unternehmung (Optimal Expansion Paths of the Firm). Gabler, Wiesbaden.